RESEARCH STATEMENT

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Mathematical research has been my passion since my early high school days. I got to dive deeper in mathematics when I started training for the International Mathematical Olympiad, and this made me explore mathematical thinking and problem solving from early on. In my college years, I approached both pure and applied sides of mathematical thinking by pursuing Electrical Engineering and Mathematics. This made discover and appreciate the interactions between pure and applied mathematics, and triggered my interest in different branches on mathematics and their interconnections.

During my doctorate degree and my early career research, I got to go back to my passions in mathematics: description, understanding, problem solving, and communicating. I like to describe phenomena using mathematics, from real situations to patterns in abstract structures. Seeking that description provides a way to better understanding, and in my opinion, this is the core of mathematical thinking. By a deeper understanding, we are capable to formulating questions that we want to solve and this motivates creativity. Finally, I enjoy communicating these ideas and discoveries, not only to colleagues, but also to everyone.

Introduction

My current research focuses on spectral zeta functions, heat kernels, Laplace-type operators, asymptotic analysis, and their applications to regularization problems in Quantum Field Theory and Number Theory. Specifically, I have been studying their relationship with Quantum Field Effects, with a particular emphasis on the Casimir Effect, regularization of divergent series, and asymptotic behavior of sequences appearing in Number Theory.

My work has been focused on finding the analytic continuation of spectral zeta functions that arise from the study of differential operators defined on compact manifolds subject to different boundary conditions. This methodology provides an efficient way of suppressing the presence of undefined series and infinities that often come up in the study of quantum field theories. Also, I have been interested in the study of asymptotic number theory by analyzing asymptotic properties of sequences with number theoretical characteristics.

I am also interested in finding connections between the geometry of manifolds, boundary conditions, and their influence in the resulting spectral zeta function. Studying the impact of these characteristics on the eigenvalues and their distribution is essential to achieve a better understanding of spectral functions and their physical implications. Among other interests that I have is the study of densities and asymptotic behavior number sequences, specifically related to prime numbers.

Background

The Casimir effect is a quantum field effect result of the zero-point energy fluctuations in a quantum mechanical system [28, 6]. Given a Hamiltonian H, the eigenvalue problem associated with the system,

$$H\Psi = \lambda\Psi \tag{1}$$

gives the fundamental frequencies at which the system resonates. Quantizing the harmonic oscillator shows that the zero-point energy is given by

$$E = \frac{\hbar\lambda}{2}\,, \tag{2}$$

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for λ the fundamental frequency of the oscillator. Therefore, it follows the formal definition of the Casimir energy

$$E = \frac{\hbar}{2} \sum_{\lambda \in \sigma(H)} \lambda \,, \tag{3}$$

for $\sigma(H)$ the set of eigenvalues of H [28, 21].

In a physical context, most of the Hamiltonians arise as differential operators acting on the space $L^2(M)$ of finite energy wave functions defined on a manifold. In order to have a consistent theory, it is essential to have a self-adjoint differential operator that depends on the geometry of the manifold and the boundary conditions. Given H to be a self-adjoint operator acting on a Hilbert space, there is a characterization of its spectrum given by [8, 34],

Theorem 1 (Spectrum Self-Adjoint Operator)**.** *If H is a self-adjoint operator acting on a Hilbert space, then its point spectrum is real, discrete and with no limit point besides infinity.*

This result states that it is possible to label the eigenvalues as $\{\lambda_n\}_{n=1}^{\infty}$ and they are such that $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \leq \ldots$ and $\lambda_n \to \infty$ as $n \to \infty$. Because of this, Equation (3) is not going to be well defined, as the expression diverges. In order to avoid such inconvenience, regularization techniques are used to obtain finite results. For this purpose, it is common [28, 21, 36, 19, 10] to use the help of the spectral zeta functions defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-2s} \,, \tag{4}$$

where the λ_n are taken to be distinct from zero. We have that this expression will converge for sufficiently large $\Re(s)$ [40, 21, 10, 36]. To formally obtain the expression for the Casimir Energy, we can consider the value $s = -1/2$ on the spectral zeta function. As $s = -1/2$ is not in the region of convergence of ζ, complex analysis techniques are used to find a meromorphic continuation of the spectral function to a region containing $s = -1/2$. This technique is known as Zeta Function Regularization, which is widely used to regularize divergent expressions arising from physical systems [28, 21, 36, 19, 10].

Traditionally, regularizing expressions using the zeta function scheme requires explicit knowledge of the eigenvalues of the operator. However, finding the actual eigenvalues for a configuration is, in general, not possible. Hence, one cannot directly calculate the spectral zeta function nor its analytic continuation for more than just a limited number of cases.

Recently, a method has been developed to overcome this problem, to find the zeta function and its analytic continuation without explicitly knowing the eigenvalues [21, 23]. To achieve this purpose, one can make use of *Cauchy's Residue Theorem*, which states that [37],

Theorem 2 (Cauchy's Residue Theorem)**.** *Given F a meromorphic function and a close curve γ, then*

$$\frac{1}{2\pi i} \int_{\gamma} F(z) dz = \sum_{a \in A} I(\gamma, a) Res\, F(z)|_{z=a}$$

where A is the set of poles of F inside γ, and I is the winding number of γ around a.

By finding a suitable function F whose zeros are the eigenvalues of H, and by applying these previous results, it is possible to find an expression for the zeta function. The characteristic equation that the eigenvalues satisfy can be obtained by imposing the boundary conditions of the problem. If $f(\lambda)$ is such that $f(\lambda) = 0$ gives the eigenvalues of H, then one can consider the function

$$F(\lambda) = \lambda^{-2s} \frac{f'(\lambda)}{f(\lambda)} \tag{5}$$

which is meromorphic in the entire complex plane and has poles at the eigenvalues of H with

$$\operatorname{Res} F(\lambda)|_{\lambda=\lambda_n} = \lambda_n^{-2s} \,. \tag{6}$$

In order to apply Cauchy's Residue Theorem to this function and capture all of the eigenvalues some regularity conditions of F at infinity are required, as an infinite contour must be considered. This provides an integral representation for the zeta function, which is well suited for performing its analytic continuation. By subtracting asymptotic terms of F as $\lambda \to \infty$, it is possible to extend the convergence region of the integral representation of the zeta function further to the left. This procedure can be continued to include any point in the complex plane, in particular to a region containing $s = -1/2$ [21, 23, 10].

Calculations of the spectral zeta function are hence deeply connected to the geometry of the underlying manifold and the boundary conditions applied, although a direct method to describe such connections is still elusive at this point[16, 2].

This method involving an integral representation and its analytic continuation of the spectral zeta function provides a path to investigate other properties of these differential operators, such as the determinant of the operator, which gives information about the one loop contribution of the effective action [21, 19, 41, 18].

Moreover, other important spectral functions, such as the heat kernel, can be related to the spectral zeta function in a geometric way. Information regarding geometric properties of the underlying space such as the heat kernel coefficients can be related with the poles of the spectral zeta function associated to the space. Hence, this method provides a very efficient way of finding such information without the need of appealing to other functions[21, 28, 2].

Other type of information encoded in the spectral zeta function relates to the cohomology of the ambient manifold. The de Rham cohomology as well as the Reidemeister torsion are examples of such information also encoded in the spectral zeta function [33].

Current and Past Research

For my doctorate work, I explored the zeta functions arising from a *Piston* configuration in different types of manifolds and boundary conditions. Basically, a piston is the result of joining two configurations by a common boundary and boundary conditions, such that this boundary is movable in a normal direction [11, 21, 14, 13]. By finding the resulting spectral zeta function of this new setting, it is possible to find the self-energy of the system without having undefined quantities or infinities.

The self energy is then dependent on the position of the piston, creating a force when the piston is not fixed and is allowed to move in the system. Hence, the general interrogation is to determine the dynamics of the piston due to the quantum vacuum, and possibly, external fields.

The research that has been made in this area is still dependent on the specific geometry of the manifold and to the boundary conditions in a unknown manner, resulting in completely different behaviors for the piston when the boundary conditions are changed [21, 13, 14, 31].

In my work, I revisited the classical calculations made in the context of Dirichlet or Neumann boundary conditions at the piston for different configurations, but I also explored new approaches where I considered boundaries modeled by potentials and distributions. The idea behind this is to generalize the treatment of boundary conditions which have proven to be pivotal to the behavior of the quantum vacuum fluctuations [12, 14, 13, 31]. In pursuing this approach, I studied smooth and non-smooth distributions that played the role of a piston for different types of systems. [31].

The technical difficulties that arise from this approach is the treatment of asymptotic behavior for the eigenvalues and eigenfunctions, since explicit expressions are generally not available. Classical asymptotic analysis techniques such as WKB expansions, stationary phase methods, and in general, multiple scale analysis. With these methods, it is possible to obtain the leading asymptotic terms that provide the analytic continuation of the spectral zeta function, without explicit knowledge of certain parameters, such as a potential or perturbation function.

Using this strategies, I found the Casmir Energy, Casimir Force, and Functional Determinant for several configurations by finding the analytic continuation for the corresponding spectral zeta functions which included the special points $s = 0, -1/2$ [12, 3, 27, 30].

Besides providing a methodology to obtain expressions for the analytic continuation of the spectral zeta function, utilizing asymptotic expansions coming from large parameter asymptotics are well suited to perform numerical calculations. Understanding this problem has led to results involving the universal coefficients of the heat kernel expansion in terms of the geometric properties of the underlying manifold [3, 12, 30].

I have being attracted to studying the influence of symmetries and boundary conditions on the behavior of quantum vacuum fluctuations. Another project I made was to analyze the effect on small surface perturbations on the change in the Casimir energy [29]. The Casimir is greatly influenced by the local geometry [21, 28, 7]. One of the most interesting questions in the field is to analyze what is the dependence on the geometry and boundary conditions. By analyzing perturbations on surfaces of revolution, I seek to obtain better understanding of these implications. The understanding of the geometric dependence of the Casimir effect is of great importance nowadays, as nano technology is getting closer and closer, and a deeper understanding of the quantum world is imminent for the development of new technologies. This study can clarify the geometric preference for quantum systems for a specific type of curvature and boundary conditions.

Currently I am working on generalizing my results on perturbed surfaces of revolution to higher dimensions by considering warped manifolds $M = I \times_f N$, where I is an interval, N is a compact Riemannian manifold, and f is the warping function. Here I consider the effects of a perturbation $f \mapsto f + \epsilon g$ on the zeta function associated with the Laplacian on M. This shows to provide well defined quantities for even dimension manifolds but to have a different behavior in odd dimensions.

In order to better understand the behavior contour integrals appearing in integral representations of zeta functions, I am also studying the effects of taking contour integrals of one forms over an essential singularity. This is important as many zeta functions despite not having a pole at infinity will present an essential singularity at infinity that will contribute when computing contour integrals over paths that go to infinity, specially over vertical lines. These integrals will present a Stokes-phenomenon-like behavior, in which there are different results depending the angle in which the contour passes through the essential singularity.

Future Plans

The study of spectral functions continues to gain increased appreciation throughout various branches of mathematics and physics. Because of the sheer amount of geometric information these functions encode, this field not only provides an attractive area of study, but also a potentially fertile area of work. Currently, many connections between such functions, the spectrum of differential operators, and the geometry of the underlying space together with the boundary conditions remain to be discovered and explored.

As is presently known, there can be spaces with different geometric structures that give rise to identical spectral properties. The existence of non-isometric isospectral manifolds provided a starting point to investigate which of the geometric properties contribute to the behavior of the spectrum of these operators [16].

One of the goals of the field is to provide a way to analyze the behavior of the spectral functions, by a direct characterization in terms of geometric properties of the domain and the boundary properties imposed.

My research has been directed towards finding a general way of finding spectral zeta functions independently of the boundary conditions and the geometry of the space. I have been doing this by studying broader classes of potentials in the setup of Casimir pistons.

My research plans cover the further study of the effect of perturbations on zeta functions. Not only perturbations involving the manifold but also the effect of perturbing boundary conditions. By this I seek to discover the main factors that influence the behavior of zeta functions as well as the change at special values, for example the Casimir energy at $s = -1/2$ and the functional determinant at $s = 0$.

I plan to study the effect of transformations in the metric on the zeta function and its special values. Using conformal transformations is possible to identify invariants in the structure of zeta functions and in the calculation of special values. In the case of the Casimir energy, a reverse approach can also be made. By seeking to minimize the Casimir force acting on a space, one can define a Casimir flow that changes the metric on the space in order to reduce the force.

The study of the asymptotic behavior of zeta functions and functions defining them is of prime importance. I plan to continue my research on the effect of essential singularities on contour integrals that appear in integral representations of zeta functions. This study will also provide tools to handle divergent series appearing in number theory in order to regularize them.

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